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On the Dilatational Stability of the Earth.

By Lord Rayleigh, O.M., Pres. R.S.

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The theory of elastic solids usually proceeds upon the assumption that the body is initially in a state of ease, free from stress and strain. Displacements from this condition, due to given forces, or vibrations about it, are then investigated, and they are subject to the limitation that Hooke's law shall be applicable throughout and that the strain shall everywhere be small. When we come to the case of the earth, supposed to be displaced from a state of ease by the mutual gravitation of its parts, these limits are transgressed; and several writers[*] who have adopted this point of view have indicated the obstacles which inevitably present themselves. In his interesting paper[†] Professor Jeans, in order to attain mathematical definiteness, goes the length of introducing forces to counteract the self-gravitation: "That is to say, we must artificially annul gravitation in the equilibrium configuration, so that this equilibrium configuration may be completely unstressed, and each element of matter be in its normal state." How wide a departure from actuality is here implied will be understood if we reflect that under such forces the interior of the earth would probably be as mobile as water.

It appears to me that a satisfactory treatment of these problems must start from the condition of the earth as actually stressed by its self-gravitation, and that the difficulties to be faced in following such a course may not be so great as has been supposed. The stress, which is so enormous as to transcend all ordinary experience, is of the nature of a purely hydrostatic pressure, and as to this surely there can be no serious difficulty. After great compression the response to further compressing stress is admittedly less than at first, but there is no reason to doubt that the reaction is purely elastic and that the material preserves its integrity. At this point it may be well to remark, in passing, upon the confusion often met with in geological and engineering writings arising from the failure to distinguish between a one-dimensional and a three-dimensional, or hydrostatic, pressure. When rock or cast iron is said to be *crushed* by such and such a pressure, it is the

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[*] See, for example, Love, 'Theory of Elasticity,' § 127; Chree, 'Phil. Mag., vol. 32, p. 233, 1891; Jeans, 'Phil. Trans.,' A, vol. 201, p. 157, 1903.

[†] *Loc. cit.*, p. 161.

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former kind of pressure which is, or ought to be, meant. There is no evidence of crushing under purely hydrostatic pressure, however great.

Not only is the integrity of a body unimpaired by hydrostatic pressure, but there is reason to think that the superaddition of such a pressure may preserve a body from rupture under stresses that would otherwise be fatal. FitzGerald raises this question in a review* of Hertz's 'Miscellaneous Papers.' He writes: "In considering the cracking of a material like glass, Hertz seems to think its cracking will depend only on the tension; that it will crack where the tension exceeds a certain limit. He does not seem to consider whether it might not crack by shearing with hardly any tension. It is doubtful whether a material in which there were sufficient general compression to prevent any tension at all, would crack. Rocks seem capable of being bent about and distorted to almost any extent without cracking, and this might very well be expected if they were at a sufficient depth under other rocks to prevent their parts being under tension. It is an interesting question whether a piece of glass could be bent without breaking if it were strained at the bottom of a sufficiently deep ocean. On the other hand, there seems very little doubt that the parts of a body might slide past one another under the action of a shear, and would certainly crack unless there were a sufficiently great compressional stress to prevent the crack; and that consequently a body might crack, even though the tensions were not by themselves sufficiently great to cause separation, and might crack where the shear was greatest, and not where the tensions were greatest."

When we reflect that pieces of lead may be made to unite under pressure when the surfaces are clean, and upon what is implied when insufficiently lubricated journals, or slabs of glass under polish, *seize*, we may well doubt whether it is possible to disintegrate a material at all when subjected to enormous hydrostatic pressure. In the words of Dr. Chree†: "The conditions under which the deep-seated materials of the earth exist are fundamentally different from those we are familiar with at the surface. The enormous pressure, and the presumably high temperature, very likely combine to produce a state to which the terms solid, viscous, liquid, as we understand them, are alike inapplicable."

A study of the mechanical operations of coining and of stamping (in recent years, I believe, much developed) would probably throw light upon this question. We know that rod or tube may be "squirted" from hot (but solid) lead. Is the obstacle to a similar treatment of harder material purely practical? In the laboratory I have experimented upon jellies of various degrees of stiffness, on the principle of suiting the material to the appliances rather than the appliances to the material. In the simplest arrangement a leaden bullet is imbedded in jelly contained in a strong glass tube which the bullet somewhat nearly fits. Although the tube stand vertical for several days, there is no appreciable descent. But if by numerous longitudinal impacts against a suitable pad the inertia of the bullet be brought into play, movements through several inches may be obtained. Here, although the deformations are very violent, there is no rupture visible, either before or behind the bullet.

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* 'Nature,' November 5, 1896; 'Scientific Writings,' p. 433.

† 'Phil. Mag.,' vol. 43, p. 173, 1897.

When an elastic body is slightly displaced from the condition of ease, the potential energy (V) is expressed by terms involving the squares and products of the displacements. If, however, we suppose given finite forces to be constantly imposed, so that the initial condition is one of strain, the case is somewhat, though not essentially, altered. It may be convenient to make a statement, once for all, in terms of generalised co-ordinates. If under the action of the forces Φ, Θ, ... the co-ordinates assume the values ϕ, θ, ... we have in terms of the potential energy of strain V,

$$\Phi = \frac{dV}{d\phi}, \qquad \Theta = \frac{dV}{d\theta}, \text{ etc.} \tag{1}$$

If the forces permanently imposed be distinguished by the suffix (0), they are connected with the corresponding values of the co-ordinates, ϕ_0, etc., by the equations

$$\Phi_0 = \frac{dV}{d\phi_0}, \qquad \Theta_0 = \frac{dV}{d\theta_0}, \text{ etc.} \tag{2}$$

This strained condition is now to be regarded as initial, and displacements from it are denoted by ascribing to the co-ordinates the slightly altered values $\phi_0 + \delta\phi$, $\theta_0 + \delta\theta$, etc. For the potential energy of strain we have

$$V - V_0 = \frac{dV}{d\phi_0}\delta\phi + \frac{dV}{d\theta_0}\delta\theta + \tfrac{1}{2}\frac{d^2V}{d\phi_0{}^2}(\delta\phi)^2 + \frac{d^2V}{d\phi_0\,d\theta_0}\delta\phi\,.\,\delta\theta + \ldots, \tag{3}$$

which is of the *first* order of the small quantities $\delta\phi$, etc. But $V - V_0$ is not now the whole potential energy. In addition to the potential energy of strain we have to include that of the steadily imposed forces, represented by the terms

$$-\Phi_0\,\delta\phi - \Theta_0\,\delta\theta - \ldots. \tag{4}$$

The whole potential energy is thus

$$V - V_0 - \Phi_0\,\delta\phi - \Theta_0\,\delta\theta - \ldots = \tfrac{1}{2}\frac{d^2V}{d\phi_0{}^2}(\delta\phi)^2 + \frac{d^2V}{d\phi_0 d\theta_0}\delta\phi.\delta\theta + \tfrac{1}{2}\frac{d^2V}{d\theta_0{}^2}(\delta\theta)^2 + \ldots, \tag{5}$$

regard being paid to (2). The total potential energy, as given by (5), is now of the second order in $\delta\phi$, etc., as is obviously required by the circumstance that the strained condition ϕ_0, etc., is one of equilibrium under the proposed forces. The coefficients of stability are $d^2\mathrm{V}/d\phi_0{}^2$, $d^2\mathrm{V}/d\phi_0 d\theta_0$, etc., and they may differ finitely from the values which obtained previously to the application of the forces Φ_0, etc.

As an example having an immediate bearing upon the matter in hand, let us consider the case of a uniform non-gravitating body originally in a state of ease. If a small hydrostatic pressure Φ act upon it, the volume ϕ changes proportionally, and the ratio gives the "compressibility" of the body in this condition. Under the action of a finite pressure Φ_0 the volume may be greatly altered, especially if the body be gaseous, but the new condition is still one of equilibrium and may be regarded as initial. The compressibility now may be quite different from before, but it may be treated in the same way as depending upon the small *change* of volume $\delta\phi$ accompanying the imposition of a small *additional* pressure $\delta\Phi$.

To those who, while accepting the usual elastic theory for bodies in a state of ease, repudiate the application to bodies subject to great hydrostatic pressure, I would suggest that liquids and solids, as we know them, are not really free from stress. In virtue of cohesional forces, there is every reason to believe, the interior of a drop of water is under pressure not insignificant even in comparison with those prevailing inside the earth, and the same may be said of a piece of steel.

The conclusion that I draw is that the usual equations may be applied to matter in a state of stress, provided that we allow for altered values of the elasticities. In general, these elasticities will not only vary from point to point, but be æolotropic in character. If, however, we suppose that the body is naturally isotropic, and that the imposed stress is everywhere merely a hydrostatic pressure, so that by pure expansion a state of ease could be attained, the case is much simpler and probably suffices for an approximate view of the condition of the earth. But although the initial state is one free of shear, we are not to conclude that the rigidity is the same as it would be without the imposed pressure. On the contrary, there is much reason to think that the rigidity would be increased. If there is any analogy to be found in a pile of mutually repellant hard spheres, it will follow that an infinite pressure will entail infinite rigidity as well as infinite incompressibility.

In the original draft of this paper I had supposed that it would be possible upon these lines to find another and a more practical basis for Professor Jeans' analysis. A correspondence with Professor Love* has, however, convinced me that this hope is destined to disappointment, and the remainder of the paper loses accordingly much of the interest which at first I felt for it. In Professor Jeans' theory, if Δ be the dilatation, so that the altered density is $\rho(1-\Delta)$, U the radial outward displacement, E the potential of a volume-distribution of density $\rho\Delta$, and a surface-distribution of density $-\rho U$, the displacements ξ, η, ζ are subject to

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* To whom I am indebted also for other corrections.

$$\rho\frac{d^2\xi}{dt^2} = (\lambda+\mu)\frac{d\Delta}{dx} + \mu\nabla^2\xi - \rho\frac{dE}{dx}, \tag{6}$$

and two similar equations relating to η, ζ. In (6) λ and μ are the elastic constants of Lamé's notation, and they relate to displacements from the *compressed* initial condition. From equations (6) we obtain, as usual,

$$\rho\frac{d^2\Delta}{dt^2} = (\lambda+2\mu)\nabla^2\Delta - \rho\nabla^2 E; \tag{7}$$

and by Poisson's equation

$$\nabla^2 E = -4\pi\gamma\rho\Delta, \tag{8}$$

γ being the constant of gravitation. Thus

$$\rho\frac{d^2\Delta}{dt^2} = (\lambda+2\mu)\nabla^2\Delta + 4\pi\gamma\rho^2\Delta, \tag{9}$$

which is Professor Jeans' equation.*

The solution of these equations is developed by Professor Jeans with the view of determining at what point instability sets in. Attention is concentrated mainly upon the solution of (9) expressed by a spherical function of order *one*, as being that which bears upon the question of the evolution of the moon.

I had intended merely to indicate a somewhat simpler treatment, following more closely the notation and method of Lamb's memoir, "On the Vibrations of an Elastic Sphere;"† but as the results so obtained do not agree with those of Jeans, it appears necessary to set forth the argument in fuller detail, so as to facilitate criticism.

If in (9) we assume that Δ is proportional to $\cos pt$, we get

$$(\nabla^2 + h^2)\Delta = 0, \tag{10}$$

where

$$h^2 = \frac{\rho(p^2 + 4\pi\gamma\rho)}{(\lambda+2\mu)}; \tag{11}$$

and the solution of (10), subject to the condition of finiteness at the centre, is

$$\Delta = (hr)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(hr) \,.\, S_n \,.\, \cos pt, \tag{12}$$

J being the symbol of Bessel's functions, and S_n a spherical surface function of order n. As is well known, $J_{n+\frac{1}{2}}$ is expressible in finite terms; in the case of $n = 0$, $(hr)^{\frac{1}{2}}J_{\frac{1}{2}}(hr)$ may be replaced in (12) by $\sin hr/hr$, a constant factor being disregarded.

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* *Loc. cit.*, p. 162.

† 'Proceedings London Mathematical Society,' vol. 13, p. 192, 1882.

Before going further it may be well to consider the particular case of a *fluid* for which $\mu = 0$. Here the solution for Δ already given suffices to solve the problem, and the condition of no pressure at the surface $(r = a)$ gives at once

$$J_{n+\frac{1}{2}}(ha) = 0, \tag{13}$$

which with (11) determines p^2 in terms γ, ρ, a and the elastic constant λ. The criterion of stability follows by setting $p = 0$. In the case of $n = 0$, where the displacements are symmetrical, $ha = m\pi$, m being an integer; and we see that equilibrium is unstable for symmetrical displacements if

$$a^2\rho^2\gamma > \tfrac{1}{4}\pi\lambda. \tag{14}$$

In general, by (8) and (10)

$$\nabla^2 E = \frac{4\pi\gamma\rho}{h^2}\nabla^2\Delta, \tag{15}$$

so that

$$E = \frac{4\pi\rho\gamma}{h^2}\Delta + e, \tag{16}$$

where e satisfies throughout the sphere

$$\nabla^2 e = 0. \tag{17}$$

Substituting the value of E in (6), we get with regard to (11)

$$(\nabla^2 + k^2)\,\xi = \left(1 - \frac{k^2}{h^2}\right)\frac{d\Delta}{dx} + \frac{\rho}{\mu}\,\frac{de}{dx}, \tag{18}$$

where
$$k^2 = p^2\rho/\mu. \tag{19}$$

Equation (18) and its companions may be treated as in Lamb's classical paper. A solution is

$$\xi = -\frac{1}{h^2}\,\frac{d\Delta}{dx} + \frac{1}{p^2}\,\frac{de}{dx}, \quad \text{etc.}, \tag{20}$$

where Δ satisfies (10). In virtue of (17) these values satisfy the relation

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = \Delta;$$

and the solution may be completed by the addition of terms u, v, w, satisfying $(\nabla^2 + k^2)\,u = 0$, etc., as well as the relation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Professor Lamb gives the general values of u, v, w. For our present purpose, and with limitation to one order of spherical harmonics, it suffices to take

$$u = \psi_{n-1}(kr)\frac{d\phi_n}{dx} - \frac{n}{n+1}\,\frac{k^2 r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\frac{d}{dx}\,\frac{\phi_n}{r^{2n+1}}, \tag{21}$$

and two similar equations, where ϕ_n is a solid harmonic of degree n; $r = \sqrt{(x^2+y^2+z^2)}$; and ψ_n is defined by the equation

$$\psi_n(\theta) = 1 - \frac{\theta^2}{2\,.\,2n+3} + \frac{\theta^4}{2.4.2n+3\,.\,2n+5} - \ldots. \tag{22}$$

Save as to a constant multiplier $\theta^n\psi_n(\theta)$ is identical with $\theta^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\theta)$, as employed in (12). ψ_n is thus associated with *solid* in place of *surface* harmonics. The function possesses the following properties

$$\psi_n'(\theta) = -\frac{\theta}{2n+3}\psi_{n+1}(\theta); \tag{23}$$

$$\psi_n(\theta) + \frac{\theta}{2n+1}\psi_n'(\theta) = \psi_{n-1}(\theta); \tag{24}$$

$$\psi_n(\theta) - \psi_{n-1}(\theta) = \frac{\theta^2}{2n+1\,.\,2n+3}\psi_{n+1}(\theta). \tag{25}$$

A formula in spherical harmonics frequently required is

$$x\phi_n = \frac{r^2}{2n+1}\left\{\frac{d\phi_n}{dx} - r^{2n+1}\frac{d}{dx}\,\frac{\phi_n}{r^{2n+1}}\right\}. \tag{26}$$

The term of the nth order in Δ is thus

$$\Delta_n = \psi_n(hr)\,.\,\omega_n \tag{27}$$

and corresponding thereto

$$\xi = -\frac{1}{h^2}\,\frac{d\Delta_n}{dx} + \frac{1}{p^2}\,\frac{de_n}{dx} + u, \tag{28}$$

where u is defined as above, and e_n, as well as ϕ_n and ω_n, is a solid harmonic of degree n.

The formation of the boundary conditions to be satisfied at the free surface of the sphere ($r = a$) proceeds almost exactly as in Lamb's investigation (p. 199), the only difference arising from the fact that h has now a different value. The first of the three symmetrical surface conditions may be written

$$\frac{\lambda}{\mu}x\Delta_n + \left(r\frac{d}{dr} - 1\right)\xi + \frac{d}{dx}(x\xi + y\eta + z\zeta) = 0. \qquad (29$$

The terms in (29) depending on the parts of ξ, η, ζ which involve Δ_n are found to be

$$A_n\frac{d\omega_n}{dx} + B_n\frac{d}{dx}\,\frac{\omega_n}{r^{2n+1}}, \tag{30}$$

where

$$\mathrm{A}_n = \frac{\lambda+2\mu}{\mu}\,\frac{a^2\psi_n(ha)}{2n+1} - \frac{2n-2}{h^2}\,\psi_{n-1}(ha); \tag{31}$$

$$\mathrm{B}_n = -\frac{\lambda+2\mu}{\mu}\,\frac{a^{2n+3}}{2n+1}\psi_n(ha) + \frac{2(n+2)\,a^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(ha). \tag{32}$$

In like manner Lamb finds for the terms in (29) arising from u, v, w,

$$\mathrm{C}_n\frac{d\phi_n}{dx} + \mathrm{D}_n\frac{d}{dx}\,\frac{\phi_n}{r^{2n+1}}, \tag{33}$$

where

$$\mathrm{C}_n = -\left\{\frac{k^2a^2}{2n+1}\,\psi_n(ka) - 2(n-1)\,\psi_{n-1}(ka)\right\}; \tag{34}$$

$$\mathrm{D}_n = -\frac{n}{n+1}\,\frac{k^2a^{2n+3}}{2n+1}\left\{\psi_n(ka) + \frac{2(n+2)}{k^2a^2}\,ka\,\psi_n'(ka)\right\}. \tag{35}$$

We have now further an additional part arising from e_n, which, it should be observed, makes no contribution to Δ_n. In this

$$\left(r\frac{d}{dr}-1\right)\frac{de_n}{dx} = (n-2)\frac{de_n}{dx},$$

$$x\frac{de_n}{dx} + y\frac{de_n}{dy} + z\frac{de_n}{dz} = n\,e_n;$$

so that the additional part is

$$\frac{2n-2}{p^2}\,\frac{de_n}{dx}. \tag{36}$$

The two most important cases where $n = 0$ and $n = 1$ are also especially simple, in that (36) disappears. It will be convenient to consider them first. When $n = 0$, u, v, w vanish: also, since e_0 is constant, (28) reduces to

$$\xi = -\frac{1}{h^2}\,\frac{d\Delta_0}{dx}, \qquad \eta = -\frac{1}{h^2}\,\frac{d\Delta_0}{dy}, \qquad \zeta = -\frac{1}{h^2}\,\frac{d\Delta_0}{dz}, \tag{37}$$

where Δ_0 is proportional to $\psi_0(hr)$. The motion is everywhere purely radial. Exactly as in Lamb's investigation of vibrations without gravity, the expression (30) reduces to

$$\mathrm{B}_0\frac{d}{dx}\,\frac{\omega_0}{r},$$

where ω_0 is a constant, so that the surface conditions yield simply $\mathrm{B}_0 = 0$, or from (32)

$$\frac{\lambda+2\mu}{\mu}\,\psi_0(ha) = \tfrac{4}{3}\psi_1(ha). \tag{38}$$

Writing θ for ha, and for ψ_0 and ψ_1 $(= -3\theta^{-1}\psi_0')$ their values, we get

$$\tan\theta = \frac{4\theta}{4-\theta^2\dfrac{\lambda+2\mu}{\mu}}. \tag{39}$$

Except for a slight difference of notation, this is the same as Lamb's equation, and his results are therefore available. They are expressed by means of Poisson's elastic constant σ, and they exhibit ha/π as dependent on σ and on the order of the root. To adapt them it is only necessary to remember that h^2, as given by (11), has here a different value from that which obtains when there is no gravitation ($\gamma = 0$). On the other hand, although γ be finite, ha/π may still be equated to T_1/τ, where T_1 is the time occupied by a plane wave of longitudinal vibration in traversing a space equal to the diameter of the sphere, and τ denotes the time of complete oscillation. The following are the smallest values of ha/π corresponding to selected values of σ, as given by Lamb:—

$\sigma = 0.$	$\sigma = \frac{1}{4}.$	$\sigma = \frac{3}{10}.$	$\sigma = \frac{1}{3}.$
0·6626	0·8160	0·8500	0·8733

For example, if $\sigma = \frac{1}{4}$ (Poisson's value), the criterion of stability is

$$\frac{4\pi\gamma\rho^2a^2}{\lambda+2\mu} < (0{\cdot}8160)^2.$$

If $\sigma = \frac{1}{2}$, the material is incompressible, and motion of the kind now under contemplation is excluded.

When $n = 1$, (36) again vanishes, though for a different reason from before.* The form of the solution is accordingly the same as if there were no gravitation. We have from (31), (32)

$$A_1 = \frac{\lambda+2\mu}{\mu}\,\frac{a^2\psi_1(ha)}{3}, \tag{40}$$

$$B_1 = -\frac{\lambda+2\mu}{\mu}\,\frac{a^5\psi_1(ha)}{3} + \frac{2a^5\psi_2(ha)}{5}; \tag{41}$$

and from (34), (35) omitting some common factors which have no effect,

$$C_1 = \psi_1(ka); \tag{42}$$

$$D_1 = \frac{a^3}{2}\left\{\psi_1(ka) + \frac{6}{k^2a^2}ka\psi_1'(ka)\right\} = \frac{a^3}{2}\left\{\psi_1(ka) - \tfrac{6}{5}\psi_2(ka)\right\}. \tag{43}$$

The surface conditions (29) are of the form

$$A_1\frac{d\omega_1}{dx} + B_1\frac{d}{dx}\frac{\omega_1}{r^3} + C_1\frac{d\phi_1}{dx} + D_1\frac{d}{dx}\frac{\phi_1}{r^3} = 0,$$

and, as Professor Lamb shows, they require that

$$A\omega_1 + C_1\phi_1 = 0; \qquad B_1\omega_1 + D_1\phi_1 = 0. \tag{44}$$

It follows that ϕ_1 and ω_1 must be of the same form, and also that

$$B_1/A_1 = D_1/C_1, \tag{45}$$

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* The terms de_1/dx, etc., in (28) denote in this case a uniform displacement, as of a rigid body, and naturally contribute nothing to the surface condition.

in which the values of A_1, B_1, C_1, D_1 are to be substituted from (40), (41), (42), (43). We find

$$\frac{4\mu}{\lambda+2\mu}\,\frac{\psi_2(ha)}{\psi_1(ha)} = 5-2\,\frac{\psi_2(ka)}{\psi_1(ka)}; \tag{46}$$

or if, in accordance with (23), we replace $\psi_2(\theta)$ by $-5\,\theta^{-1}\psi_1{}'(\theta)$,

$$-\frac{4\mu}{\lambda+2\mu}\,\frac{\psi_1{}'(ha)}{ha\,.\,\psi_1(ha)} = 1+\frac{2\psi_1{}'(ka)}{ka\,.\,\psi_1(ka)}. \tag{47}$$

Except for the different value of h, this agrees with Professor Lamb's equation (87).

In equation (46) there is no limitation upon the value of p. If to find the criterion of stability we put p or k equal to zero,

$$\psi_2(ka) = \psi_1(ka) = 1,$$

and the equation reduces to

$$\psi_2(ha) = \frac{3(\lambda+2\mu)}{4\mu}\,\psi_1(ha). \tag{48}$$

The equation may also be written in terms of the Bessel functions. The relation between J and ψ is

$$J_{\frac{1}{2}}(x)\times\surd(\tfrac{1}{2}\pi x) = x\,\psi_0(x) \qquad = \sin x,$$

$$J_{\frac{3}{2}}(x)\times\surd(\tfrac{1}{2}\pi x) = \tfrac{1}{3}x^2\psi_1(x) \quad = x^{-1}\sin x-\cos x.$$

$$J_{\frac{5}{2}}(x)\times\surd(\tfrac{1}{2}\pi x) = \tfrac{1}{15}x^3\psi_2(x) = (3x^{-2}-1)\sin x-3x^{-1}\cos x\,;$$

so that in terms of J's (48) becomes

$$\frac{\lambda+2\mu}{\mu}\,J_{\frac{3}{2}}(x) = \frac{20}{3x}\,J_{\frac{5}{2}}(x)\,; \tag{49}$$

or, if we introduce the circular functions,

$$\frac{(3-x^2)\tan x-3x}{x^2\tan x-x^3} = \tfrac{3}{20}\,\frac{\lambda+2\mu}{\mu}. \tag{50}$$

Unfortunately (49) does not agree with the result given by Professor Jeans. In his notation, when $n = 1$,

$$y_1\ (\text{from } 59) = -\tfrac{13}{15}\,\frac{\rho a^2}{\mu}, \qquad y_2\ (\text{from } 60) = +\tfrac{1}{30}\,\frac{\rho a^2}{\mu}\,;$$

and (54), (57), (58) give

$$\frac{\lambda+2\mu}{\mu}\,J_{\frac{3}{2}}(x) = \frac{156}{25x}\,J_{\frac{5}{2}}(x). \tag{51}$$

The comparison of processes is rendered difficult by the occurrence of several errors (possibly misprints) in Professor Jeans' paper. Thus (33) does not seem correct, and (41), (42) do not follow from (38), (39). Starting from Professor Jeans' equations just mentioned and making use of his (30), (43), (44), (45), (48), (49), I have obtained a result in harmony with my equation (49).

From (50) it is easy to calculate the value of λ/μ corresponding to any value of x. When $\mu = 0$, $\tan x = x$, of which the first root is

$$x = 1{\cdot}4303\pi = 4{\cdot}493.$$

This gives an angle of $77\frac{1}{2}°$ ($+180°$). Calculating for angles of 60°, 50°, 40°, we find

λ/μ	∞	3·840	2·056	1·221
x	4·493	4·189	4·014	3·840

It seems that the value of x is not very sensitive to variation of μ, and for such values of the ratio of λ/μ as are likely to occur, especially under high pressure, we might almost content ourselves with the fluid solution ($\mu = 0$).

The simplicity of the cases so far considered, viz., $n = 0$ and $n = 1$, depends upon our having escaped the necessity of determining the value of e_n. For values of n greater than unity this function remains in the equations, which now demand a more elaborate treatment. From (20), (21)

$$\xi = -\frac{1}{h^2}\frac{d\Delta_n}{dx} + \frac{1}{p^2}\frac{de_n}{dx} + u, \tag{52}$$

in which the second term becomes infinite when $p = 0$. In order to balance this, ϕ_n in (21) must be made infinite of the order p^{-2} or k^{-2}. Thus writing $k^{-2}\Phi_n = \phi_n$, we have

$$\begin{aligned}\frac{1}{p^2}\frac{de_n}{dx} + u = \frac{\rho}{\mu k^2}\frac{de_n}{dx} + \frac{1}{k^2}\left\{1 - \frac{k^2r^2}{2\,.\,2n+1}\right\}\frac{d\Phi_n}{dx} \\ - \frac{n}{n+1}\,\frac{r^{2n+3}}{2n+1\,.\,2n+3}\left\{1 - \frac{k^2r^2}{2\,.\,2n+5} + \dots\right\}\frac{d}{dx}\frac{\Phi_n}{r^{2n+1}}.\end{aligned}$$

Thus, as in the theory of differential equations with equal roots, we have when $k^2 = 0$,

$$\frac{1}{p^2}\frac{de_n}{dx} + u = \frac{df_n}{dx} + \frac{\rho}{\mu}\,\frac{r^2}{2\,.\,2n+1}\,\frac{de_n}{dx} + \frac{\rho}{\mu}\,\frac{n}{n+1}\,\frac{r^{2n+3}}{2n+1\,.\,2n+3}\,\frac{d}{dx}\frac{e_n}{r^{2n+1}}. \tag{53}$$

with two similar equations, f_n denoting again a solid harmonic of degree n. From (52), (53) we find for the radial displacement

$$\begin{aligned}\mathrm{U} &= \frac{x\xi + y\eta + z\zeta}{r} = -\frac{1}{h^2}\frac{d\Delta_n}{dr} + \frac{nf_n}{r} + \frac{\rho}{\mu}\,\frac{n\,r\,e_n}{2\,.\,2n+3} \\ &= \frac{\omega_n}{h}\left\{-\frac{n}{ha}\psi_n(ha) + \frac{ha}{2n+3}\psi_{n+1}(ha)\right\} + \frac{nf_n}{a} + \frac{\rho}{\mu}\,\frac{n\,a\,e_n}{2\,.\,2n+3}\end{aligned} \tag{54}$$

at the surface, where $r = a$.

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The boundary condition (29) requires a parallel treatment. The terms depending on Δ_n remain as in (30), (31), (32). From (34), (35), we get as appropriate for the present purpose

$$\mathrm{C}_n = (2n-2)\left\{1-\frac{k^2a^2}{2\,.\,2n+1}\right\}-\frac{k^2a^2}{2n+1}=2n-2-\frac{n\,k^2a^2}{2n+1},$$

$$\mathrm{D}_n = \frac{n\,k^2a^{2n+3}}{n+1\,.\,2n+1\,.\,2n+3}.$$

Equations (33), (36) now give, Φ being written as before for $k^2\phi$,

$$(2n-2)\left\{\frac{\rho}{\mu k^2}\frac{de_n}{dx}+\frac{1}{k^2}\frac{d\Phi_n}{dx}\right\}-\frac{na^2}{2n+1}\frac{d\Phi_n}{dx}+\frac{n\,a^{2n+3}}{n+1\,.\,2n+1\,.\,2n+3}\frac{d}{dx}\frac{\Phi_n}{r^{2n+1}}, \tag{55}$$

which, when k^2 is made to vanish, is to be replaced by

$$(2n-2)\frac{df_n}{dx}+\frac{\rho}{\mu}\frac{na^2}{2n+1}\frac{de_n}{dx}-\frac{\rho}{\mu}\frac{na^{2n+3}}{n+1\,.\,2n+1\,.\,2n+3}\frac{d}{dx}\frac{e_n}{r^{2n+1}}. \tag{56}$$

This is additional to (30). The equations to be satisfied at the surface are thus

$$\mathrm{A}_n\omega_n+(2n-2)f_n+\frac{\rho}{\mu}\frac{na^2}{2n+1}e_n = 0. \tag{57}$$

$$\mathrm{B}_n\omega_n-\frac{\rho}{\mu}\frac{na^{2n+3}}{n+1\,.\,2n+1\,.\,2n+3}e_n = 0. \tag{58}$$

When $n = 1$, f_n disappears, and the final condition is found by eliminating the ratio $\omega_n : e_n$ from (57), (58). This would conduct us again to the results already arrived at for that case. In general we require another equation connecting e_n with ω_n and f_n.

For this purpose we must recur to the definition (16) of e_n and of E. A calculation is made by Jeans on the basis of the expression (12) of Δ by means of Bessel's functions. We have at the surface

$$e_n = \frac{4\pi\rho\gamma a}{2n+1}\left\{-\mathrm{U}-(n+\tfrac{1}{2})\,h^{-1}(ha)^{-\frac{3}{2}}\mathrm{J}_{n+\frac{1}{2}}(ha)\,.\,\mathrm{S}_n-h^{-1}(ha)^{-\frac{1}{2}}\mathrm{J}_{n}{}'_{+\frac{1}{2}}(ha)\,.\,\mathrm{S}_n\right\}. \tag{59*}$$

In order to express this in our present notation (by means of ψ's), we see by comparison of (12) and (27) that

$$\mathrm{J}_{n+\frac{1}{2}}(ha)\,.\,\mathrm{S}_n = (ha)^{\frac{1}{2}}\,\psi_n(ha)\,.\,\omega_n, \tag{60}$$

so that with use of (24)

$$e_n = \frac{4\pi\rho\gamma a}{2n+1}\left\{-\mathrm{U}-\frac{2n+1}{h^2a}\psi_{n-1}(ha)\,.\,\omega_n\right\}. \tag{61}$$

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* In Professor Jeans' equations (5), (23) the sign of U is positive, but this appears to be an error.

Eliminating U between this and (54), we find

$$f_n = \frac{\rho a^2 e_n}{\mu}\left\{-\frac{(2n+1)\,\mu}{n\,(\lambda+2\mu)\,h^2a^2}-\frac{1}{2\,.\,2n+3}\right\}$$
$$+\frac{\omega_n}{h^2}\left\{-\frac{2n+1}{n}\,\psi_{n-1}+\psi_n-\frac{h^2a^2}{n\,.\,2n+3}\,\psi_{n+1}\right\}. \qquad (62)$$

The substitution for f_n in (57) now gives

$$\omega_n\left[\mathrm{A}_n+\frac{2n-2}{h^2}\left\{-\frac{2n+1}{n}\,\psi_{n-1}+\psi_n-\frac{h^2a^2}{n\,.\,2n+3}\,\psi_{n+1}\right\}\right]$$
$$+\frac{\rho a^2 e_n}{\mu}\left[-\frac{(2n-2)\,(2n+1)\,\mu}{n\,(\lambda+2\mu)\,h^2a^2}-\frac{2n-2}{2\,.\,2n+3}+\frac{n}{2n+1}\right]=0. \qquad (63)$$

This equation and (58) determine two values of $\omega_n : e_n$, and the elimination of this ratio gives the required final result. We will write (63) for brevity as

$$\mathrm{F}a^2\omega_n+\mathrm{G}\,(\rho a^2/\mu)\,e_n = 0, \qquad (64)$$

where by (31), and reduction with the aid of (25),

$$\mathrm{F} = \frac{\lambda+2\mu}{\mu}\,\frac{\psi_n}{2n+1}+\frac{2n-2}{h^2a^2}\left\{-\frac{2n+1}{n}\,\psi_n+\frac{h^2a^2}{2n+1\,.\,2n+3}\,\psi_{n+1}\right\}, \qquad (65)$$

$$\mathrm{G} = -\frac{(2n-2)\,(2n+1)\,\mu}{n\,(\lambda+2\mu)\,h^2a^2}-\frac{2n-2}{2\,.\,2n+3}+\frac{n}{2n+1}. \qquad (66)$$

Similarly, if (58) be written

$$\mathrm{H}a^2\omega_n+\mathrm{K}\,(\rho a^2/\mu)\,e_n = 0, \qquad (67)$$

we may take

$$\mathrm{H} = -\frac{\lambda+2\mu}{\mu}\,\frac{\psi_n}{2n+1}+\frac{2\,(n+2)}{2n+1\,.\,2n+3}\,\psi_{n+1} \qquad (68)$$

$$\mathrm{K} = -\frac{n}{n+1\,.\,2n+1\,.\,2n+3}; \qquad (69)$$

and the final result is

$$\mathrm{FK}-\mathrm{GH} = 0, \qquad (70)$$

giving the ratio $\psi_{n+1}(ha) : \psi_n(ha)$ in terms of n, ha, and $(\lambda+2\mu)/\mu$.

In applying results of calculation based upon the assumption of a uniform compressibility to the case of the earth where the variation is likely to be very considerable, we must have regard to the character of the function (12) by which the dilatation is expressed. When $n = 1$ or a greater number, (12) vanishes at the centre and (when $\mu = 0$) at the surface. The values to be ascribed to the elasticities are those proper to an intermediate position, such as half-way between the centre and the surface. For a more complete treatment we might calculate the balance of the elastic and gravitational potential energies on the basis of a displacement still following the same law as has been found to apply to a uniform sphere. In accordance with a general principle the result, so calculated, will be correct as far as the first powers of the variations from uniformity.

Another question, interesting to geologists, upon which our results have a bearing is as to the effect of denudation in altering the surface level. The immediate effect of the removal of material is, of course, to lower the level, but if the material removed is heavy and the substratum very compressible, the springing up of the foundation may more than neutralize the first effect and leave the new surface higher than the old one. So far as I am aware discussions have been based upon the elastic quality merely of the interior without regard to self-gravitation; but, as is easy to see, if the condition be one approaching instability, the effect of a pressure applied to the surface may be immensely increased.

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